Chapter 5

The Structure of Reason Relations

> Hi group,
> This is a very, very rough and incomplete draft of a chapter in which I basically want to explain inferential role semantics in philosophical terms, while also offering a compare and contrast with truth-maker semantics. This is really just a first collection of ideas, but I think it would help me to talk about them.
> Ulf

The previous chapter presented a theory of the structure that we have found in the norms governing assertions and denials, on the one hand, and truth-makers and falsity-makers, on the other hand. We have characterized this structure at a very abstract level. In this chapter, we will connect this abstract description back to our philosophical concerns.

5.1 Inferential Role Semantics and Truth-Makers Semantics

The inferential role semantics from the previous chapter and the truth-maker semantics from Chapter 3 have a lot in common. In both semantic theories, the interpretatants of sentences are pairs of sets. In inferential role semantics, they are pairs of premisory and conclusory roles, which are sets of implications (i.e. pairs of premises and conclusions) to which the sentence can be added as premise or as conclusion, respectively, to make a good implication. In truth-maker semantics, they are pairs of states, namely the sentence's truth-makers and falsity-makers.

We can formulate the semantic clauses for the logical connectives in the same way for both semantic theories. To see this, let's write $[\![\phi]\!] = \left\langle |\phi|^+, |\phi|^- \right\rangle$ for the pair of sets that is the interpretant of ϕ, we can state the clauses for the logical connectives thus:

AND $\quad [\![\phi \& \psi]\!] = \left\langle |\phi|^+ \Cup |\psi|^+, |\phi|^- \cap |\psi|^- \right\rangle$

OR $\quad [\![\phi \vee \psi]\!] = \left\langle |\phi|^+ \cap |\psi|^+, |\phi|^- \Cup |\psi|^- \right\rangle$

IF $\quad [\![\phi \rightarrow \psi]\!] = \left\langle |\phi|^- \cap |\psi|^+, |\phi|^+ \Cup |\psi|^- \right\rangle$

NOT $\quad [\![\neg\phi]\!] = \left\langle |\phi|^-, |\phi|^+ \right\rangle$

Here "$\cap$" is set-theoretic intersection in both, inferential role semantics and truth-maker semantics. However, we must interpret "$\Cup$" differently in the two theories. In truth-maker theory, we must read it as pair-wise fusion of states, so that $|\phi|^x \Cup |\psi|^x$ (with x being either $+$ or $-$) is the set of states that are fusions of an element of $|\phi|^x$ and an element of $|\psi|^x$, where $|\phi|^+$ is the set of ϕ's truth-makers and $|\phi|^-$ is the set of ϕ's falsity-makers. So, in the notation of Chapter 3, e.g., $|\phi|^+ \Cup |\psi|^-$ is $\{s \sqcup t \mid s \Vdash \phi \text{ and } t \dashv\!\!| \phi\}$.

In inferential role semantics, however, we must read "$\Cup$" as giving us the (premisory or conclusory) inferential role of the addition-to-a-sequent of the two sentences whose roles we are combining, where the superscript "$+$" means that the sentence must be added on the left and the superscript "$-$" means that the sentence must be added on the right. So, for example, $|\phi|^+ \Cup |\psi|^+$ is the inferential role of "ϕ, ψ" on the left of a sequent, i.e., the sets of implications such that we can add "ϕ, ψ" as premises and get a good implication.[1] In both cases, $\Cup$ is a way to combine parts of interpretants to further things that can serve as such parts of interpretants.

The inferential role semantics interpretation of these clauses brings out their close link to sequent rules. For each connective, the first element of the pair in its clause corresponds to the left rule and the second element of the pair corresponds to the right rule. When we see a "$\Cup$" or just elements of interpretants, we have a one-premise sequent rule. And in cases where we see a "$\cap$", the corresponding sequent rule has two premise sequents, each of which corresponds to one of the sets that are intersected. Finally, if we have "$|A|^+$", then "A" occurs on the left side of a premise sequent. And if we have "$|A|^-$", then "A" occurs on the right side of a premise sequent. Contexts are always shared among all sequents in a rule.

To see what this means, let's look at some examples. The part of AND that corresponds to the conjunction right rule is the second element in the pair that occurs in AND, that is: $|\phi|^- \cap |\psi|^-$. So, our conjunction right rule has two premise sequents, and ϕ and ψ occur on the right in these sequents respectively. Given that contexts are shared, this means that our right conjunction rule allows us to infer from $\Gamma \Rightarrow \phi, \Delta$ and $\Gamma \Rightarrow \psi, \Delta$ to $\Gamma \Rightarrow \phi \& \psi, \Delta$. For another example, consider the conditional right rule. It

[1] Formally: $|\phi|^+ \Cup |\psi|^+$ is $\{\langle x \cup y, z \cup w\rangle \mid \langle x, z\rangle \in (|\phi|^+)^\vee, \langle y, w\rangle \in (|\psi|^+)^\vee\}^\vee$; in words, it is the inferential role of the set of pair-wise unions of the complement inferential roles of ϕ and ψ. To see why this makes sense, note that $(|\phi|^+)^\vee$ is the set of sequents that are equivalent to $\phi \Rightarrow$ in how they combine to form good implications, and similarly for $(|\psi|^+)^\vee$. So, $|\phi|^+ \Cup |\psi|^+$ is the set of pairs, $\langle \Gamma, \Delta\rangle$, that can be added to $\phi, \psi \Rightarrow$ to get a good implication, i.e., such that $\Gamma, \phi, \psi \Rightarrow \Delta$ is a good implication.

corresponds to $|\phi|^{+} \Cup |\psi|^{-}$. So we have a one-premise sequent rule, and ϕ occurs on the left and ψ on the right in the premise sequent. Hence, our rule allows us to infer from $\Gamma, \phi \Rightarrow \psi, \Delta$ to $\Gamma \Rightarrow \phi \rightarrow \psi, \Delta$. Finally, the negation left rule corresponds to $|\phi|^{-}$. So it is a one-premise sequent rule with ϕ on the right. Hence, it allows us to infer from $\Gamma \Rightarrow \phi, \Delta$ to $\Gamma, \neg\phi \Rightarrow \Delta$. It is easy to see that this works for all rules.

In light of these correspondences, it is clear that the four clauses above—AND, OR, IF, and NOT—capture part of the structure that is shared between the sequent calculus of Chapter 2, the truth-maker semantics of Chapter 3, and the inferential role semantics of Chapter 4. Indeed, the inferential role semantics of Chapter 4 already gave us an abstract characterization of this common structure. However, we want to understand this common structure not only mathematically but also philosophically, and we want to understand how the realizations of this structure might differ from each other. In understanding the common structure and how its realizations might differ, we will clarify the nature of reason relations themselves. This is the goal of the current chapter.

5.1.1 Two Differences Between the Approaches

When we look at our three formal theories that share the structure captured by AND, OR, IF, and NOT, we can see at least two important differences. First, the sequent calculus differs—at least on the most straightforward reading—from truth-maker semantics and inferential role semantics in its restrictions to finitely many premises and conclusions and finite derivations. Second, truth-maker semantics differs from the sequent calculus and inferential role semantics in its appeal to worldly states that swing free of the sentences of our language. Let's look at these two differences in turn.

We typically think of sequents as containing only finitely many premises and finitely many conclusions, and we think of sequent derivations as having finitely many steps. It are, of course, these "finitistic" strictures that typically lead to divergences between proof theory and model theory. Model theory typically doesn't build in any restrictions to finite premise sets or the like, and this leads to model theories appearing more powerful. However, we can often do in proof theoretic settings, in effect, what we do in model theory by lifting the usual "finitistic" strictures. Take, for example, arithmetic. If we allow ourselves to add the ω-rule, which has infinitely many premises, to Peano Arithmetic, then we reach True Arithmetic.[2]

I'm unsure about a lot of this. There are tricky mathematico-logical questions here.

Something similar holds for our case here. If we allow premises, conclusions, and derivations in our sequent calculus to be infinite, then there is no difference in power between our three formal theories. For our purposes, nothing hangs on premises, conclusions, or derivations being finite. Hence, we will allow for infinite premise-sets, conclusion-sets, and sequent derivations wherever necessary. As a result, the difference between our sequent calculus and truth-maker semantics isn't of any special significance for us here. It is merely a particular instance of the general and familiar difference between proof theoretic approaches that insist on certain things being finite and model theoretic approaches that don't build in any such constraints.

The second difference is of much greater importance to us. In our sequent calculus and inferential role semantics, we specify what follows from what in a logically complex language by appeal to what follows from what in the language without logical vocabulary. And the interpretants of our sentences are set-theoretic constructions out of our sentences, namely a pair of (premisory and a conclusory role, i.e.) sets of (sequents, i.e.) pairs of sets (or multisets) of sentences. In truth-maker semantics, by contrast, we specify what follows from what by appeal to which fusions of states are

[2] Of course, Peano Arithmetic with the ω-rule has its own Gödel sentence(s). But that is also true of any model theory.

possible or impossible. And the interpretants of our sentences are pairs of sets of states, namely truth-makers and falsity-makers. Thus, the sequent calculus and inferential role semantics, as it were, never leave the space of things of which they give an account, namely sentences or, perhaps, discursive acts. In contrast to this, truth-maker semantics appeals to what is represented by sentences in order to given an account of the reason relations between these sentences. This can look like a striking and crucial difference between truth-maker theory and inferential role semantics. And we should, hence, have a closer look at this difference.

Before we turn to such a closer look, however, let us foreshadow the view of the matter that we will endorse. We hold that in order for their to be a genuine difference here, this difference must show up in the consequence relations or meanings that the two semantic theories can capture. Now, it seems that there is indeed such a difference in the meanings that the two theories can capture. Truth-maker theory can assign different meanings (interpretants) to sentences that must receive the same meaning (interpretant) by inferential role semantics. This difference arises from the requirement in inferential role semantics that we are dealing only with *proper* inferential roles, i.e. inferential roles, IR, such that $IR = IR^{\vee\vee}$. Insofar as this requirement is justified and ought to be enforced in truth-maker semantics, the two theories are equivalent. We will argue that the insistence on proper inferential roles is indeed justified because meanings cannot cut more finely than inferential roles.

5.1.2 Inferentially Equivalent Meanings?

In order to consider the second difference from the previous subsection in more detail, we must first bring it into sharper focus. To that end, it will prove useful to construct a case that can occur in truth-maker theory but not in inferential role semantics.

In truth maker-semantics, there can be a model with two distinct states, s and t, such that for all other states, x, the fusion $s \sqcup x$ is possible iff the fusion $t \sqcup x$ is possible. The states s and t play the same role with respect to the possibility and impossibility of states of which they are parts. Moreover, we can have sentences S and T such that the only truth-maker of sentence S is state s and the only truth-maker of sentence T is state t. And we can assume that the same is true for their distinct falsity-makers, which we may label $\bar{s}$ and $\bar{t}$. So that $\bar{s} \sqcup x$ is possible iff the fusion $\bar{t} \sqcup x$ is possible. So the interpretants of our two sentences are: $[\![S]\!] = \langle \{s\}, \{\bar{s}\} \rangle$ and $[\![T]\!] = \langle \{t\}, \{\bar{t}\} \rangle$. These are clearly distinct interpretants. In this sense, S and T differ in meaning in this scenario.

Notice, however, that our two sentences have the exact same inferential role, i.e., for all Γ and Δ, we have $\Gamma, S \models_{TM} \Delta$ iff $\Gamma, T \models_{TM} \Delta$ and we have $\Gamma \models_{TM} S, \Delta$ iff $\Gamma \models_{TM} T, \Delta$. Moreover, two complex sentences such that one is the result of substituting S for T somewhere will also have the same inferential roles, i.e., the contributions of S and T to the inferential roles of complex sentences in which they occur are identical. Hence, our consequence relation cannot distinguish between S and T, except by the fact that they are different signs. The question that we will face in a moment is whether we should allow such differences in meaning without any difference in inferential role, and our position will be that we shouldn't allow this.

But before we put forward our view, consider how the situation is different in inferential role semantics. Recall that in inferential role semantics, we require that inferential roles are proper inferential roles, and we say that the inferential role of sentence ϕ is proper iff $[\![\phi]\!] = [\![\phi]\!]^{\vee\vee}$. What this means is that for every sentence, σ, such that, for all $\langle \Gamma, \Delta \rangle$, we have $\Gamma, \phi \models_{IR} \Delta$ iff $\Gamma, \sigma \models_{IR} \Delta$ and we have $\Gamma \models_{IR} \phi, \Delta$ iff $\Gamma \models_{IR} \sigma, \Delta$, the interpretants of σ and ϕ are identical, i.e., $[\![\phi]\!] = [\![\sigma]\!]$. After all, if, for all $\langle \Gamma, \Delta \rangle$, we have $\Gamma, \phi \models_{IR} \Delta$ iff $\Gamma, \sigma \models_{IR} \Delta$ and we have $\Gamma \models_{IR} \phi, \Delta$ iff $\Gamma \models_{IR} \sigma, \Delta$, then,

Here I need a way to mark phase semantics consequence. I use the subscript IR below the turnstile.

by definition, $[\![\phi]\!]^{\vee} = [\![\sigma]\!]^{\vee}$. Hence, $[\![\phi]\!]^{\vee\vee} = [\![\sigma]\!]^{\vee\vee}$, and by the requirement that the inferential role of ϕ be proper it follows that $[\![\phi]\!] = [\![\sigma]\!]^{\vee\vee}$. In other words, the requirement that all sentences must have proper inferential roles is, in effect, the requirement that two sentences with the same inferential role must be assigned the same interpretant. Or again: meanings cannot cut more finely than inferential roles.

The upshot of what we just said is this: that interpretants in truth-maker semantics are states that can vary independently of the consequence relation while this is not so for the interpretants in inferential role semantics gives rise, in truth-maker semantics, to the possibility of sentences with the same inferential role but different interpretants while no such possibility exists in inferential role semantics. This difference between the two theories is, as it were, the "cash value" of the difference that, unlike inferential role semantics, truth-maker semantics appeals to the worldly states represented by sentences. This is not just a technical result that holds for the two formal theories that we have considered. It is plausibly the crucial philosophical difference between representationalist and inferentialist semantic theories that according to the first there can be sentences with the same inferential roles that differ in meaning while, according to inferentialism, we cannot distinguish meanings more finely than inferential roles.

Why do we think that meanings cannot cut more finely than inferential roles? This is one of the places where our pragmatics-first approach is important. What we mean by a sentence is a matter—we hold—of the norms that govern our use of the sentence, in particular assertions and denials of the sentence in the context of giving reasons for or against claims. Now, consequence relations are the formal objects that we use to theorize reason relations, i.e., the relations that hold between when one claim (or more) is a reason for or against another. If a consequence relation captures the norms that govern the use of two sentences and these two sentences don't differ in their role in this consequence relation, then the use of these two sentences is governed by the same norms. It follows that the two sentences mean the same.

An opponent would have to hold that two sentences can differ in meaning without there being any difference in how the two sentences can be used correctly. Such a distinction between two meanings would be a distinction without a difference. If whether we use one or the other sentence never makes any normative difference to our rational discourse, then any difference between the sentence's meanings can at most be of the sort that Frege called a difference in "coloring," i.e., a difference that may be important in poetry but that doesn't deserve to be called a difference in meaning.

5.1.3 Can Truth-Makers Do More than Inferential Roles?

In response to the argument just given, an opponent might point out that our argument is only as strong as the reasons to accept a conception of meaning on which what we mean is a matter of the norms that govern the use of our expressions. And the opponent might go on to argue that this conception of meaning will come under pressure if it turns out that truth-makers and falsity-makers can do useful theoretical work that inferential roles cannot do. For such a difference may lend support to a view of meaning on which representational content of an expression can differ without any difference in the norms that govern the use of the expression.

We have two responses to this worry. The first response is abstract and general; it says that what we mean by "meaning" is something that can be individuated by reason relations. These reason relations are the topic that we are investigating. So, given our topic and what we mean by "meaning," it is not an open possibility that we must make distinctions that don't make any difference for consequence relations. Any consequence relation, however, that can be represented in truth-maker semantics can also be represented in inferential role semantics (see the Appendix for the proof of this result).

Proposition 1. *There is a truth-maker model in which* $\Gamma \mathrel{|\!\!\!\equiv_{TM}} \Delta$ *iff there is an inferential role model in which* $\Gamma \mathrel{|\!\!\!\equiv_{IR}} \Delta$.

In what way should we mark that this is Bob's result?

Ignoring the details of the proof, this holds because what varies between the models is the material base consequence relation, and both frameworks can represent any such relation over sets of atomic sentences. So the two theories are equivalent in their ability to codify consequence relations. Hence, there cannot be a reason for us to think that truth-maker semantics can capture important differences that inferential role semantics cannot capture.

Our second response is much more detailed and long-winded. It consists in showing, in the next subsection, how we can capture accounts of philosophically interesting notions that have been developed in truth-maker theory within our framework. It will prove useful, however, to (a) use our sequent calculus for this and (b) use Correia's (2016) version of truth-maker theory. Regarding (a), we will use the sequent calculus and not directly the inferential role semantics because the results are easier to comprehend and prove in this way. Given how closely our sequent calculus and our inferential role semantics are connected, this is merely a matter of presentation. Regarding (b), Correia's version of truth-maker semantics allows for failures of distribution that can also arise in our theories. Correia (2016) gives philosophical arguments for allowing such failures of distribution. We agree with him and, hence, we will use Correia's rather than Fine's version of truth-maker theory. We will start by considering a version of what Fine calls "analytic equivalence" that Correia developed and calls "factual equivalence." We will then turn to Correia-style variants of Fine's notions of containment, entailment, and subject matter.

Is this true? Could we do it in phase semantics?

5.1.4 Recapturing Truth-Maker Accounts

Correia is interested in determining when two sentences, A and B, describe the same facts or states in virtue of their (propositional) logical form, written $A \approx B$. Correia's (2016) logic of this relation is a proper fragment of Angell's (1989; 1977) logic of analytic entailment, which Fine (2016) advocates as a logic of content. Correia's logic differs from Angell's logic in that it doesn't validate the distributive principle according to which $A \vee (B \wedge C)$ is equivalent to $(A \vee B) \wedge (A \vee C)$.[3] We start by showing how this logic can be recovered from our sequent calculus.

Correia provides two semantic characterizations and a Hilbert-style axiomatisation of his logic. The Hilbert-style system includes the following axiom (using Correia's label).

A10 $\quad A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$

We get Angell's first-degree system for analytic equivalence by adding the distribution principle that Correia rejects as another axiom, namely:

A11 $\quad A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$

If instead of adding A11 to our axioms, we replace A10 with A11, we get what Correia calls his "dual system."

Correia proves of the system with just A10 that it is sound and complete with respect to equivalence in truth-maker theory in the following sense:

Fact 2. *$A \approx B$ is a theorem of Correia's logic iff A and B have the same truth-makers in every truth-maker model (Correia, 2016).*

[3] Formally speaking, the fact that we agree with Correia on this point means we will not insist on propositions being what Fine calls "convex," i.e., we will not insist that every state that has a verifier of the proposition as a part and that is itself part of a verifier of the proposition is also a verifier of the proposition. This constraint is what allows Fine to adopt Angell's logic rather than Correia's logic, i.e., what allows him to endorse the distributive principle in A11.

Since truth-makers and falsity-makers vary independently of $S^{\Diamond}$ across models, this result holds independently of the constraints we put on possible states. Hence, our operational sequent rules should suffice to capture Correia's factual equivalences. Indeed, they do so in the following way:

Proposition 3. *$A \approx B$ is a theorem of Correia's logic iff the operational rules of* NMMS *suffice to show that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B \succ \Delta$ and vice versa are admissible (appendix: Proposition 15).*

Given the soundness and completeness of NMMS with respect to inferential role semantics, this implies that $A \approx B$ is a theorem of Correia's logic iff A and B have the same premisory inferential roles in all inferential role models, including models in which CO fails (but permutation and contraction hold). This means that two sentences have the same truth-makers in all models just in case, in virtue of their logical form, they play the same role as premises, i.e., they are inter-substitutable as premises *salva consequentia* in virtue of their logical form and independently of structural rules (other than permutation and contraction).

Correia (2016, 117) shows that $A \approx B$ is provable in his dual system iff $\neg A \approx \neg B$ is provable in his original system. Given the negation sequent rules, $\Gamma, \neg A \succ \Delta$ is derivable iff $\Gamma \succ A, \Delta$ is derivable, and $\Gamma, \neg B \succ \Delta$ is derivable iff $\Gamma \succ B, \Delta$ is derivable. This implies

Dan, is that correct?

Proposition 4. *$A \approx B$ is a theorem of Correia's dual logic iff the operational rules of* NMMS *suffice to show that the moves from $\Gamma \succ A, \Delta$ to $\Gamma \succ B, \Delta$ and vice versa are admissible iff A and B have the same falsity-makers in all models.*

Hence, factual equivalence in Correia's dual system holds between sentences just in case the sentences play, in virtue of their logical form, the same inferential role as conclusions. In inferential role semantics terms: the two sentences have the same conclusory role in all models, including models in which CO fails (but where permutation and contraction hold).

Let us now move on to recover a variant of Fine's notion of containment, modulo the distribution principle A11 above.[4] Fine defines containment as follows (Fine, 2017a, 640-41).

Definition 5. *Containment*: A contains B iff (i) every verifier of A includes as a part a verifier of B and (ii) every verifier of B is included as a part in a verifier of A.

Since containment is defined purely in terms of verifiers, it should be recoverable as concerning the left side of sequents. Indeed, we can recover containment in virtue of logical form as follows.

Proposition 6. *A contains B, in virtue of logical form, iff, for some* Θ*, the operational rules of* NMMS *suffice to show that the moves from* $\Gamma, A \succ \Delta$ *to* $\Gamma, B, \Theta \succ \Delta$ *and vice versa are admissible (appendix: Proposition 19).*

Thus the containment holds just in case A and $\{B\} \cup \Theta$ play the same inferential roles as premises, so that this inferential role of A as a premise has a part that is the inferential role of B as a premise. In terms of inferential role semantics, A contains B in virtue of logical form iff, in all models (including those where CO (but not permutation and contraction) fails), A has the same role as a set of sentences that includes B. Besides containment, Fine often uses the notion of entailment, which he defines as follows (Fine, 2017a, 640-41).

Definition 7. *Entailment*: A entails B iff every verifier of A is a verifier of B.

The following result allows us to recast entailment in virtue of logical form in terms of our sequent rules.

[4] As already intimated, my notions differ from Fine's insofar as he requires propositions to be convex.

Proposition 8. *A entails B, in virtue of logical form, iff the operational rules of* NMMS *suffice to show that the move from* $\Gamma, B \succ \Delta$ *to* $\Gamma, A \succ \Delta$ *is admissible (appendix: Proposition 21).*

Thus, entailment orders propositions by their strength as premises. Formulating this in inferential role semantics yields: A entails B, in virtue of logical form, iff in all models (including ones in which CO (but not permutation and contraction) fails) the premisory role of B is a subset of the premisory role of A. In other words, merely in virtue of the meaning of logical vocabulary, in every good implication in which B occurs as a premise, we can replace B by A and we still have a good implication.

Let us end by giving a characterization of what Fine calls the "subject matter" of a proposition (Fine, 2017b, 697).

Definition 9. *Subject-matter*: The subject-matter of a bilateral proposition is a pair in which the first element is the fusion of all of its verifiers and the second element is the fusion of all of its falsifiers.

If we do a proof-search on $A\succ$, we get a set of atomic sequents such that the union of all the states that any of these sequents deems impossible are exactly the truth-makers of A (appendix: Proposition 12). And since the falsity-makers of A are the truth-makers of $\neg A$, a proof-search on $\neg A\succ$ and hence, on $\succ A$ yields a set of atomic sequents that together deem exactly the falsity-makers of A impossible. So, it is obvious that we can characterize subject-matter as follows:

Proposition 10. *Given the proof-trees that result from proof-searches on* $A\succ$ *and* $\succ A$*, the subject-matter of the proposition that A expresses is the pair,* $\langle \boldsymbol{a}, \boldsymbol{a'} \rangle$*, where* $\boldsymbol{a}$ *is the fusion of states deemed impossible by the leaves of the proof-tree for* $A\succ$*, and* $\boldsymbol{a}'$ *is this fusion for* $\succ A$*.*

Putting this in terms of inferential role semantics, this means that the subject matter of a sentence A is the pair of the adjunction of its premisory and the adjunction of its conclusory role.

Dan, is that correct? I find it hard to think through what a proof-search is in inferential role semantics. There should be something in here about atomic sentences, I guess.

This concludes our demonstration of how we can recapture accounts of factual equivalence, entailment, containment, and subject matter that were developed in truth-maker theory without appealing to worldly states that are represented by sentences. Since it are precisely accounts like these that an opponent might have thought are available in truth-maker theory but not in inferential role semantics, our recapture of these accounts provides a second response to the worry that truth-makers might be able to do more theoretical work than inferential roles. Given that our topic are reason relations as encoded in consequence relations and our commitment to meanings being individuated by roles in such relations, we find it very satisfying that we can recapture notions like factual equivalence and subject matter purely in terms of consequence relations.

5.1.5 Taking Stock

The upshot of this section is that inferential role semantics captures everything in truth-maker theory that is worth capturing. The treatment of logical vocabulary in the two theories is isomorphic, and they can represent exactly the same consequence relations. The main difference between the two theories is that, in truth-maker theory, sentences can have different interpretants while sharing their role in the consequence relation. This cannot happen in inferential role semantics. We have argued that this difference doesn't point to any philosophically meaningful advantages of truth-maker semantics over inferential role semantics, for two reasons: First, what we mean by a sentence is a matter of the norms that govern the use of the sentence, which are encoded in the consequence relation. Hence, meanings cannot be more finely individuated than inferential roles. Second, the philosophical accounts of factual equivalence, entailment, containment, and subject matter that one might think depend on worldly states that are individuated more finely than inferential roles can be recaptured in inferential role semantics.

If what we have said so far is correct, then inferential role semantics captures the essential structure of reason relations. We don't miss out on anything of philosophical importance by eschewing appeal to worldly states in our theory of reason relations. However, we still need a better understanding of the philosophical significance of the structure of reason relations as revealed by inferential role semantics. That will be the topic of the next section.

5.2 The Philosophical Significance of the Structure of Reason Relations

In the previous section, we have satisfied ourselves that there is no philosophically significant structure in truth-maker semantics that is not also in inferential role semantics. And in the previous chapter, we saw that this is also the structure of our sequent calculus. Thus, we have arrived at a view of what the structure of reason relations is. Mathematically speaking it is a set of pairs that forms a commutative monoid with a distinguished subset. While this is an interesting claim, it doesn't seem immediately philosophically illuminating. That the mathematical structure of reason relations is such a monoid plus distinguished subset stands in need of philosophical elucidation and interpretation. In this section, we will start to provide such an elucidation and interpretation.

It will turn out that there are three interrelated, philosophically crucial aspects to the mathematical structure of reason relations. First, there is the fact that the basic elements of the structure are pairs; these are candidate implications. Second, there is the commutative monoidal structure of the set of these pairs. That is, the adjunction operation for these pairs is associative and commutative, and $\langle \emptyset, \emptyset \rangle$ is an identity element. Third, there is the distinguished subset of the monoid set, which is the set of good implications (which corresponds to the set of impossible states in truth-maker

Dan, is that correct? Or should we use the monoid of inferential roles, i.e. sets of pairs of sets of pairs of sentences as our elements (instead of sets of pairs of sentences)? The latter would have the nice consequence that our monoid set is the set of meanings, and we semantics). And this distinguished subset swings free of the monoidal structure, in a sense that we will explain below. We will discuss the philosophical significance of these three aspects of inferential role semantics in turn.

5.2.1 The Importance of Having Two Sides

That the basic elements in the structure of reason relations are pairs is an expression of the fact that reason relations hold among the kind of contents that can be asserted or denied, the kind of content that can be true or false regarding how things are.[5] It is essential to these contents that they have two sides; one side that corresponds to being asserted or made true, and another side that corresponds to being denied or made false. Whatever stands in reason relations to other things, we want to argue, must be two-sided in this way.

Let's start by clarifying the two sides a bit more. The contents that stand in reason relations are the kind of contents with respect to which we can ask: "Right or wrong?", "Yes or no?" You can say, e.g.: "It is raining outside. Right or wrong?" And then I may say "Yes, correct" or "No, that's wrong" or I can remain silent on the matter. And when you wonder whether such a response of mine is correct, you can look at a part or aspect of the world and consider whether this part or aspect of the world renders my response correct or not. Indeed, what it is to be part of the world is, plausibly, to be the kind of thing that can render assertions and denials correct or incorrect. And, of course, a part or aspect of the world can also leave such issues open. What's going on in my fridge, e.g., by-and-large leaves issues about the weather outside open.

[5]While there may be analogues of reason relations among the contents of questions, imperatives, and the like, this is not our topic in this book. We think these issues are downstream of reason relations among contents that can be asserted and denied, and so we set these issues aside here.

Now, such contents are what they are in virtue of the relations among them. More specifically, some combinations of positive and negative sides of contents are incompatible. An assertion or truth-maker of "Ane is a donkey" is incompatible with an assertion or truth-maker of "Ane is a fish" and it is also incompatible with a denial or falsity-maker of "Ane is a mammal." What it is to be a content is to stand in such relations. And what it is to be a specific such content is to stand in specific such relations.

Reason relations are such relations of incompatibility. They are modal exclusion relations among contents. Something is a reason for a content just in case it excludes the negative side of the content, and something is a reason against a content just in case it excludes the positive side of the content. Given this notion of reasons for and against, there can be reasons for and against contents only if contents have two sides.

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5.2.2 The Unimportance of Order

The second important feature of the structure of reason relations, as revealed by inferential role semantics, is the structure of a commutative monoid. That means that the pairs that are the basic elements of our theory form a set with an operation on them that is associative and commutative, and the set includes an identity element. In inferential role semantics, the operation is adjunction and the identity elements is $\langle \emptyset, \emptyset \rangle$. The philosophical significance of this is that it reflects the fact that the members of the pairs that are our basic elements are sets (or multisets), so that their ordering doesn't matter. This irrelevance of the order has nice consequences for our account of logical vocabulary, and these nice consequences are what matters philosophically.

Let's start by being very clear what it means for inferential role models to be commutative monoids (ignoring for now the distinguished subset that will occupy us in the next subsection). The pairs with which we start can of course be collected into a set. What makes this a commutative monoid is a commutative and associative operation with an identity element. So, let's look at these three features of the operation that, in our case, is adjunction. (a) Adjunction is associative because $(\langle\Gamma,\Delta\rangle \uplus \langle\Theta,\Xi\rangle) \uplus \langle\Pi,\Sigma\rangle$ is the same as $\langle\Gamma,\Delta\rangle \uplus (\langle\Theta,\Xi\rangle \uplus \langle\Pi,\Sigma\rangle)$, namely $\langle\Gamma \cup \Theta \cup \Pi, \Delta \cup \Xi \cup \Sigma\rangle$. (b) Adjunction is commutative because $\langle\Gamma,\Delta\rangle \uplus \langle\Theta,\Xi\rangle$ is the same as $\langle\Theta,\Xi\rangle \uplus \langle\Gamma,\Delta\rangle$, namely $\langle\Gamma \cup \Theta, \Delta \cup \Xi\rangle$. (c) The pair $\langle\emptyset,\emptyset\rangle$ is an identity element for adjunction because $\langle\Gamma,\Delta\rangle \uplus \langle\emptyset,\emptyset\rangle$ and $\langle\emptyset,\emptyset\rangle \uplus \langle\Gamma,\Delta\rangle$ are both the same as $\langle\Gamma,\Delta\rangle$; for $\langle\Gamma \cup \emptyset, \Delta \cup \emptyset\rangle = \langle\Gamma,\Delta\rangle$.

It is easy to see that all three features (a)–(c) are ensured by the fact that adjunction combines the members of our pairs by set union (or multiset union). In particular, (a) holds because union is associative, (b) holds because union is commutative, and (c) holds because the empty set is an identity element for union. Hence, that inferential role models have the structure of a commutative monoid is a reflection of the fact that adjunction works by taking the union of the members of our pairs.

Now, what it means for adjunction to work by taking unions of the members of our pairs is that these pairs are sets (or multisets). If, for example, the members of our basic pairs were lists, adjunction would be non-commutative, baring drastic further adjustments. And if the members were trees, adjunction would be non-associative, baring drastic further adjustments. So, when we ask why inferential role models have the structure of commutative monoids, we are really asking why our basic pairs are pairs of sets, rather than pairs of things with more internal structure. The corresponding question in truth-maker theory would be why fusion is associative and commutative (and has the null state as its identity element).

When we say that the members of our basic pairs are sets (or multisets), we are saying that contents combine as the premises or conclusions of implications in an ungrouped and unordered way. Besides the obvious virtue of simplicity, this fits nicely with the observation that the order or grouping of premises in an argument is usually considered irrelevant. The same conclusions follow, e.g., from "It is raining. Moreover, it is cold" and "It is cold. Moreover, it is raining." And the same is plausible for combinations of truth-makers and falsity-makers. It seems plausible that the way in which worldly states combine into larger worldly states is ungrouped and unordered. The combination, e.g., of the states of it raining and it being cold is identical to the combination of it being cold and it raining. And whether we first combine the state of it raining with the state of it being November and then combine this combination with it being cold or whether we add the state of it being November at the end doesn't matter; the result is the same combined state.

Besides yielding the intuitively correct results, the associativity and commutativity of the internal structure of the members of our basic pairs has nice consequences for our account of logical vocabulary. In particular, it is this irrelevance of grouping and ordering that yields the associativity and commutativity of conjunction and disjunction. That is, $\phi \& \psi$ has the same premisory and conclusory role as $\psi \& \phi$, and the same holds for $\phi \vee \psi$ and $\psi \vee \phi$. And this is so because the parts of inferential roles of conjuncts and disjuncts are combined to yield the parts of inferential roles of their conjunctions and disjunctions ultimately boil down to taking unions and intersections, which are associative and commutative.

The obvious non-commutativity of the conditional, i.e. the fact that $\phi \to \psi$ differs in meaning from $\psi \to \phi$, is ensured by the fact that its inferential role combines the premisory role of one constituent content with the conclusory role of the other. Nevertheless, the irrelevance of order makes itself felt even for the conditional. Since all the antecedents in right-nested conditionals contribute only their premisory roles to the conclusory role of the conditional and only their conclusory roles to the premisory role of the conditional, their ordering doesn't matter, i.e., $\phi \to (\chi \to \psi)$ and

$\chi \to (\phi \to \psi)$ have the same inferential role. It is again the structure of the commutative monoid—together with the clauses for the conditional of course—that lies behind this irrelevance of order.

It is only negation that is an exception here. That ϕ and $\neg\neg\phi$ have the same inferential role doesn't depend on the commutative monoidal structure of inferential role semantics. Rather, it is a reflection of the two-sidedness of our contents, which means that two flips of the elements of our pairs yield the original pair. We can observe, however, that this symmetry of pairs play nicely with the structure of the commutative monoid. For every element of our monoid is a pair and, hence, can be flipped. This guarantees that all inferential roles have negations. And flipped pairs are just pairs; they can hence be combined with other inferential roles in all the usual ways. This guarantees the existence of inferential roles of complex sentences in which there are embedded negations.

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5.2.3 Good and Bad Implications

We now turn to the third and last crucial feature of inferential role semantics that stands in need of philosophical elucidation and interpretation, namely the distinguished subsets of our set of basic pairs. In inferential role semantics, we call the members of this distinguished subset the "good implications" and all other implications can be called "bad implications" (where a bad implication is, in one sense of this term, no more an implication than a toy bicycle is a bicycle). In truth-maker semantics, the states in the distinguished subset were the impossible states. What is philosophically crucial and in need of elucidation is that our treatment of the logical connectives is, in an important sense, independent of this distinguished subset. Our aim in this subsection is to explain this.

Let's start by bringing out the independence of the treatment of the logical vocabulary and the distinguished subset in more detail. Notice that the clauses for the logical connectives that we have given at the start of this chapter do not make any essential appeal to the distinguished subset, i.e., no appeal is made to the difference between good and bad implications. This is easy to miss for inferential role semantics because inferential roles are defined in terms of the good implications in which something can figure as a premise and as a conclusion. Our semantic clauses, however, merely specify the inferential roles of complex sentences in terms of the inferential roles of their constituent sentences. And it doesn't matter for this what the good implications actually are. This is broadly similar to how the semantic clauses in classical propositional logic specify the truth-values of complex sentences in terms of the truth-values of their constituents, where this is the same in all models, independently of the truth-values of the constituents in any particular model. However, this independence goes much deeper in inferential role semantics.

It is helpful, at this point, to look at truth-maker semantics again. In truth-maker semantics there isn't even any apparent appeal to the distinction between possible and impossible states in the treatment of logical connectives. The truth-makers of conjunctions, e.g., are the fusions of a truth-maker for each conjuct. The falsity-makers are the falsity-makers of either conjunct. There is no need to mention possible or impossible states in saying this. Moreover, when you look back at Fine's definitions of entailment and containment in the previous subsection, it is easy to check that they don't bring in the notions of possible and impossible states at all. Indeed, one can use truth-maker theory without the distinction between possible and impossible states to recapture, e.g., relevance logics (Jago, 2020). And Fine sees it as a virtue of his truth-maker theory that it can be spelled out in entirely non-modal terms, with the modal notions of possible states being an optional additional element of the theory.

The same holds for inferential role semantics, except that the difference between good and bad implication is build in to the analogues of states in truth-maker semantics, so that the independence that we are pointing out here is harder to see. To see what we mean, consider that in many logics, logical consequence is defined in terms of an ordering the interpretants of sentences and the logical connectives are treated by a recursive definition of their interpretants. In classical propositional logic, e.g., our interpretants of sentences are the two truth-values; we give a treatment of logical connectives by a recursive definition of their truth-values (in each model), and we can think of consequence as defined by saying that truth is higher in the ordering of interpretants than falsity and consequence holds iff, in all models, the conclusion has an interpretant that is at least as high in the ordering as the lowest premise interpretant. The same general recipe holds, e.g., of many logics in which semantic interpretants form a lattice.

The situation is radically different in inferential role semantics.[6] Our commutative monoid gives us the structure of the interpretants of our sentences, but it is silent on what follows from what. The commutative monoid gives us an account of the structure of meanings, but it is only the distinguished subset that gives us an account of consequence.

It is this independence of the interpretants of our sentences and what follows from what that allows inferential role semantics to codify open reason relations. To see this, consider how inferential role semantics allows for failures of cut and weakening. For a failure of cut, all we need to do is to construct a model in which $\langle \Gamma, \Delta \cup \{A\} \rangle$ and $\langle \Gamma \cup \{A\}, \Delta \rangle$ are in the distinguished subset but $\langle \Gamma, \Delta \rangle$ isn't in the distinguished subset. Since consequence isn't defined in terms of an ordering among interpretants or the like, we can simply stipulate such failures of cut as we please, ultimately by making sure that the atomic implications that are sufficient for such a failure, given our treatment of the logical vocabulary, are in the distinguished subset. Similarly, we can let weakening fail by letting $\langle \Gamma, \Delta \rangle$

[6]The source of this feature of inferential role semantics is Girard's phase semantics for linear logic. But we think that the philosophical significance of this feature doesn't come out clearly in phase semantics for linear logic.

but not $\langle \Gamma \cup \{A\}, \Delta \rangle$ be in the distinguished subset. And again this swings free from our commutative monoid, as long as the monoid respects the structure required by the logical connectives.

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5.3 The Structure of Reasons and Logical Expressivism

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5.4 Conclusion

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5.5 Appendix

Proposition 11. *There is a truth-maker model in which* $\Gamma \mathrel{\models\!\!\!=_{TM}} \Delta$ *iff there is an inferential role model in which* $\Gamma \mathrel{\models\!\!\!=_{IR}} \Delta$.

Proof. Left-to-right: Beginning with a truth-maker model, one can define an inferential role model that corresponds to it in the sense of defining exactly the same implications and incompatibilities. We are given a truth-maker model of a language $\mathfrak{L}_0$, defined on a modalized state space $\langle S, S^{\Diamond}, \sqsubseteq \rangle$, which assigns to each sentence $A \in \mathfrak{L}_0$ a pair of sets of states $\left\langle |A|^+, |A|^- \right\rangle$ understood as verifiers and falsifiers of that sentence and similarly for sets of sentences. The points of the implicational phase space being defined are ordered pairs of sets of sentences of $\mathfrak{L}_0$. These are the candidate implications. What corresponds to fusion, $\sqcup$, is adjunction: $\langle \Gamma, \Delta \rangle \cup \langle \Theta, \Psi \rangle$

$= \langle \Gamma \cup \Theta, \Delta \cup \Psi \rangle$, as usually defined in implicational phase space semantics. Now, let $\mathbf{I}_0$, i.e. the set of *good* implications, be as follows:

$$\langle \Gamma, \Delta \rangle \in \mathbf{I}_0 \text{ iff } \forall s, t \in S \, (s \Vdash \Gamma \text{ and } t \dashVv \Delta, \text{ then } s \sqcup t \notin S^{\Diamond})$$

That is, $\langle \Gamma, \Delta \rangle$ is a good implication just in case the fusion of any state s that verifies all of Γ and any state t that falsifies all of Δ is an impossible state, in the truth-maker model. This construction obviously guarantees that exactly the same implications will hold in the inferential role model and in the truth-maker model.

Right-to-left: We are given an implicational phase space defined on a language $\mathfrak{L}_0$: $\langle \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0), \mathbf{I}_0 \rangle$. We now define a truth-maker model with the same consequence relation as follows: The states will be candidate implications. $S = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0)$. Fusion, $\sqcup$, is adjunction: $\langle \Gamma, \Delta \rangle \sqcup \langle \Theta, \Psi \rangle = \langle \Gamma \cup \Theta, \Delta \cup \Psi \rangle$. And the possible states are the complement of the good implications: $S^{\diamond} = S - \mathbf{I}_0$. It remains to define the valuation function, which assigns to each sentence $A \in \mathfrak{L}_0$ its truth-makers and falsity-makers. Valuations must be such that $\langle \Gamma, \Delta \rangle \in \mathbf{I}_0$ iff $\forall s, t \in S \, (s \Vdash \Gamma$ and $t \dashVv \Delta$, then $s \sqcup t \notin S^{\Diamond})$. To do this, we can now define valuations as follows: $\left\langle |A|^{+}, |A|^{-} \right\rangle$ is $\langle \langle A, \emptyset \rangle, \langle \emptyset, A \rangle \rangle$. Since fusion is defined as adjunction, this automatically generalizes to sets, such that $s \Vdash \Gamma$ iff $s = \langle \Gamma, \emptyset \rangle$ $= \langle \gamma_1, \emptyset \rangle \uplus ... \uplus \langle \gamma_n, \emptyset \rangle$ for $\Gamma = \{\gamma_1, ..., \gamma_n\}$. To show that this works, in the sense of yielding the same implications in the truth-maker model and the inferential role model, we must show that $\langle \Gamma, \Delta \rangle \in \mathbf{I}_0$ iff $\forall s, t \in S \, (s \Vdash \Gamma$ and $t \dashVv \Delta$, then $s \sqcup t \notin S^{\Diamond})$. To show the left-to-right direction: If $\langle \Gamma, \Delta \rangle \in \mathbf{I}_0$, then $|\Gamma|^{+} = \langle \Gamma, \emptyset \rangle$ and $|\Delta|^{-} = \langle \emptyset, \Delta \rangle$. Since by hypothesis $\langle \Gamma, \Delta \rangle \in \mathbf{I}_0$ and $S^{\diamond} = S - \mathbf{I}_0$, it follows that $\langle \Gamma, \Delta \rangle \notin S^{\diamond}$. Moreover $\langle \Gamma, \Delta \rangle$ is the fusion of the verifier of Γ, namely $\langle \Gamma, \emptyset \rangle$, and the falsifier of Δ, namely $\langle \emptyset, \Delta \rangle$ because it is the result of adjoining them: $\langle \Gamma, \Delta \rangle = \langle \Gamma, \emptyset \rangle \uplus \langle \emptyset, \Delta \rangle$. To show the right-to-left direction: Suppose that $\forall s, t \in S \, (s \Vdash \Gamma$ and

$t \dashv\!\| \Delta$, then $s \sqcup t \notin S^{\Diamond}$). Now, $s = \langle \Gamma, \emptyset \rangle$ and $t = \langle \emptyset, \Delta \rangle$ and their fusion is $\langle \Gamma, \Delta \rangle$. Hence, $\langle \Gamma, \Delta \rangle \notin S^{\Diamond}$. Since $S^{\diamond} = S - \mathbf{I}_0$, if follows that $\langle \Gamma, \Delta \rangle \in \mathbf{I}_0$. ∎

5.5.1 Relation of CL to Correia's Logic, Containment, and Entailment

Proposition 12. *The leaves of a proof-tree that result from a proof-search on $A \succ$ are such that the union of the states that they deem impossible is exactly the set of truth-makers of A.*

Proof. A tree that results from a proof-search uses only top-to-bottom applications of operational rules. By Lemma **??**, the union of the states deemed impossible by the top-sequents of such a rule-application is the set of states deemed impossible by the bottom sequent. Hence, for any proof-tree that results from a proof-search, the union of the states deemed impossible by the leaves of the tree is the set of states deemed impossible by the root. The set of states deemed impossible by $A \succ$ are exactly the truth-makers of A. Hence, the proposition holds. ∎

As Correia (2016) shows, the following fact holds:

Fact 13. *$A \approx B$ is a theorem of Correia's logic iff A and B have the same truth-makers in every truth-maker model.*

Proposition 14. *$A \approx B$ iff proof-searches on $A \succ$ and $B \succ$ yield the same result.*

Proof. $A \approx B$ holds iff A and B have the same truth-makers in all models. By Proposition 12, this holds just in case proof-searches in CL on $A \succ$ and $B \succ$ yield the same result. ∎

Since proof-searches use only the operational rules of CL, this last result immediately implies the following proposition, which was our target.

Proposition 15. *$A \approx B$ is a theorem of Correia's logic iff the operational rules of* CL *suffice to show that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B \succ \Delta$ and vice versa are admissible.*

Fact 16. *As Correia (2016, 117) shows, $A \approx B$ is provable in his dual system iff $\neg A \approx \neg B$ is provable in his original system.*

Proposition 17. *$A \approx B$ is a theorem of Correia's dual logic iff the operational rules of* CL *suffice to show that the moves from $\Gamma \succ A, \Delta$ to $\Gamma \succ B, \Delta$ and vice versa are admissible iff A and B have the same falsity-makers in all models.*

Proof. Suppose that $A \approx B$ is a theorem of Correia's dual logic. By Fact 16 and Proposition 15 and the negation rules of CL, this happens iff the moves from $\Gamma \succ A, \Delta$ to $\Gamma \succ B, \Delta$ and vice versa are admissible. And this happens just in case proof-searches for $\succ A$ and $\succ B$ yield the same atomic sequents. By reasoning that is parallel to the proof of Proposition 12, this happens iff A and B have the same falsity-makers in all models. ∎

Definition 18. *Containment*: A contains B iff (i) every verifier of A includes as a part a verifier of B and (ii) every verifier of B is included as a part in a verifier of A.

Proposition 19. *A contains B in virtue of logical form iff, for some Θ, the operational rules of* CL *suffice to show that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B, \Theta \succ \Delta$ and vice versa are admissible.*

Proof. Left-to-right: Suppose that A contains B. By definition, there is a proposition R such that $|A|^+ = \{b \sqcup r : b \in |B|^+ \text{ and } r \in |R|^+\}$. In accordance with Assumption **??**, let Θ be a set of sentences such that the set of fusions of verifiers for each of the elements is $|R|^+$. Hence, A and $B \wedge \bigwedge \Theta$ have the same verifiers. Therefore, by Proposition 12, a proof-search on $A\succ$ and on $B \wedge \bigwedge \Theta\succ$ yield the same result. This ensures that the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B, \Theta \succ \Delta$ and vice versa are admissible.

Right-to-left: Suppose the moves from $\Gamma, A \succ \Delta$ to $\Gamma, B, \Theta \succ \Delta$ and vice versa are admissible. This happens only if proof-searches on $A\succ$ and on $B, \Theta\succ$ yield the same result. By Proposition 12, it follows that $\{t \sqcup p : t \in |\bigwedge \Theta|^+ \text{ and } b \in |B|^+\} = |A|^+$. So every verifier of A includes a verifier of B as a part, and every verifier of B is included as a part in a verifier of A. Therefore, A contains B in virtue of logical form. ■

Definition 20. *Entailment*: A entails B iff every verifier of A is a verifier of B.

Proposition 21. *A entails B in virtue of logical form iff the operational rules of* CL *suffice to show that the move from $\Gamma, B \succ \Delta$ to $\Gamma, A \succ \Delta$ is admissible.*

Proof. Left-to-right: Suppose that A entails B in virtue of logical form and, hence, in all models. Then the verifiers of A are a subset of the verifiers of B. So, by Proposition 12, the union of the states deemed impossible by the leaves of the proof-tree for $A\succ$ is a subset of the union of the states deemed impossible by the leaves of the proof-tree for $B\succ$. And this holds in all models. Suppose for reductio that there is a leaf, $\Gamma_0 \succ \Delta_0$, in the proof-tree for $A\succ$ that is not also a leaf in the tree for $B\succ$. We take a model in which s is the unique state that is deemed impossible by $\Gamma_0 \succ \Delta_0$, and we ensure that s is not deemed impossible by any of the leaves in the tree for $B\succ$. Then s is a verifier of A but not of B, contradicting our assumption that A entails B. But if the leaves of the proof-tree for $A\succ$ is a subset of those for $B\succ$, then the move from $\Gamma, B \succ \Delta$ to $\Gamma, A \succ \Delta$ is admissible.

Right-to-left: Suppose that the move from $\Gamma, B \succ \Delta$ to $\Gamma, A \succ \Delta$ is admissible. Then proof-searches on $A\succ$ and on $B\succ$ yield proof-trees such that the leaves of the tree for $A\succ$ is a subset of the leaves of the tree for $B\succ$. This must hold in virtue of logical form. The union of the states deemed impossible by the leaves of the tree for $A\succ$ is a subset of the corresponding states for $B\succ$. By Proposition 12, every verifier of A is a verifier of B. So, A entails B in virtue of logical form. ■

Bibliography

Angell, Richard Bradshaw (1977). Three systems of first degree entailment. *Journal of Symbolic Logic*, *42*, 147.

Angell, Richard Bradshaw (1989). Deducibility, entailment and analytic containment. In R. Norman, J.and Sylvan (Ed.) *Directions in Relevant Logic*, vol. 42, (pp. 119–143). Dordrecht: Kluwer AcademicPublishers.

Correia, Fabrice (2016). On the logic of factual equivalence. *Review of Symbolic Logic*, *9*(1), 103–122.

Fine, Kit (2016). Angellic content. *Journal of Philosophical Logic*, *45*(2), 199–226.

Fine, Kit (2017a). A theory of truthmaker content I: Conjunction, disjunction and negation. *Journal of Philosophical Logic*, *46*(6), 625–674.

Fine, Kit (2017b). A theory of truthmaker content II: Subject-matter, common content, remainder and ground. *Journal of Philosophical Logic*, *46*(6), 675–702.

Jago, Mark (2020). Truthmaker semantics for relevant logic. *Journal of Philosophical Logic*, *49*(4), 681–702.